ON THE SINGULAR POINTS OF A FUNCTION
GIVEN BY ITS SERIES EXPANSION
AND
THE IMPOSSIBILITY OF ANALYTICAL PROLONGATION IN VERY GENERAL CASES*

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1. An analytic function of the imaginary variable z being given, if we develop it following the powers of $z - a$, the circle of convergence generally passes through a single singular point, the one that is closest to point a.

If, on the contrary, the coefficients of the series $\sum_0^\infty a_n z^n$ are chosen arbitrarily, in such a way however that the radius of convergence is neither null nor infinite, the function defined by this series has at least one singular point on the circle of convergence, but in general it has several. In a Memoir on functions given by their Taylor development (*Journal de Liouville*, 4th series, vol. VIII), M. Hadamard has given important results on the search for these singular points. I propose to generalize some of these results and to deduce from them new methods particularly for searching if a point on the circle of convergence is singular. In many of these methods, even all the points of the circle of convergence are singular, which allows the formation of series, much more general than those currently common, which cannot be analytically prolonged beyond the circle of convergence.

*Sur les points singuliers d'une fonction donnée par son développement en série et l'impossibilité du prolongement analytique dans des cas très généraux. *Annales scientifiques de l'É.N.S.* 3e série, tome 13 (1896), p. 367–399. Translated by V. T. Toth utilizing Claude (Anthropic, Inc.) AI technology.

2. Let

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

be a series whose circle of convergence has radius 1, that is to say, as M. Hadamard showed, such that the upper limit, for $m = \infty$, of $\sqrt[n]{|a_n|}$ is equal to 1, or that of $\frac{\mathbf{L}|a_n|}{n}$ is equal to 0. Let us form

$$\frac{f^n(t)}{1.2\ldots n} = a_n + a_{n+1} t \frac{n+1}{1} + \ldots + a_{n+p} t^p \frac{(n+1)(n+2)\ldots(n+p)}{1.2\ldots p} + \ldots,$$

where t is real and between 0 and 1.

If $z = 1$ is not a singular point of $f(z)$, the upper limit, for $n = \infty$, of $\sqrt[n]{\frac{f^{(n)}(1)}{1.2\ldots n}}$ is smaller than $\frac{1}{1-t}$. If $z = 1$ is singular, it is equal to $\frac{1}{1-t}$.

The coefficients $t^p \frac{(n+1)\ldots(n+p)}{1.2\ldots p}$ increase as long as $p < \frac{nt}{1-t}$, then decrease constantly. The largest corresponds to the value of p such that

$$\frac{nt}{1-t} - 1 \leq p < \frac{nt}{1-t}.$$

More generally, let us give p a value such that $\frac{p}{n}$ has for limit $\frac{t}{1-t}$, when n increases indefinitely.

The use of the asymptotic value of the function $\Gamma(n)$ shows that the nth root of this coefficient has for limit $\frac{1}{1-t}$. Indeed, we have,

$$\begin{aligned}
&\mathbf{L}\frac{n+p}{p} + \int_1^n \mathbf{L}\frac{n+x}{x} dx < \mathbf{L}\frac{n+1}{1} \\
&\quad + \mathbf{L}\frac{n+2}{2} + \ldots + \mathbf{L}\frac{n+p}{p} \leq \mathbf{L}(n+1) + \int_1^p \mathbf{L}\frac{n+x}{x} dx, \\
&\mathbf{L}\frac{n+p}{n+1} + \frac{p}{n}\mathbf{L}\left(\frac{n+p}{p}\right) + \frac{1}{n}\mathbf{L}\frac{n+p}{p(n+1)} \\
&\quad < \frac{1}{n}\mathbf{L}\left[\frac{(n+1)(n+2)\ldots(n+p)}{1.2\ldots p}\rho^p\right] < \mathbf{L}\frac{n+p}{n+1} + \frac{p}{n}\mathbf{L}\left(\frac{n+pt}{p}\right).
\end{aligned}$$

If p increases indefinitely with n, in such a way that $\frac{p}{n}$ has for limit $\frac{t}{1-t}$, the two extreme members have the same limit $\mathbf{L}\frac{1}{1-t}$ and $\sqrt[n]{t^p\frac{(n+1)\dots(n+p)}{1.2\dots p}}$ tends towards $\frac{1}{1-t}$.

Dividing by this coefficient, and setting $p+n=m$. Let

$$\begin{aligned}\varphi_m(t) = a_m + a_{m+1}t\frac{m+1}{p+1} + a_{m+2}t^2\frac{(m+1)(m+2)}{(p+1)(p+2)} + \dots \\ + a_{m+\nu}t^\nu\frac{(m+1)\dots(m+\nu)}{(p+1)\dots(p+\nu)} + \dots \\ + a_{m-1}\frac{1}{m}t^{-1} + \dots + a_{m-p}\frac{1}{t^p}\frac{p!(p-1)\dots 1}{m(m-1)\dots(m-p+1)},\end{aligned}$$

where p increases with m, in such a way that $\frac{p}{m}$ tends towards t.

If $z=1$ is not a singular point of $f(z)$, $\sqrt[n]{|\varphi_m(t)|}$ has an upper limit smaller than 1, and as $\frac{m}{n}$ has a positive limit, $\frac{1}{1-t}$, the same is true for $\sqrt[m]{|\varphi_m(t)|}$.

If $z=1$ is singular, the upper limit of $\sqrt[n]{|\varphi_m(t)|}$, and also that of $\sqrt[m]{|\varphi_m(t)|}$, is equal to 1 if, when m increases indefinitely, $n=m-p$ passes through all integer values; which takes place, for example, if we take the number p between mt and $mt-1$.

But if $\nu \geq \lambda m$, λ being a fixed quantity also small as we wish, it is useless to take into account the terms of $\varphi_m(t)$ that follow $a_{m+\nu}t^\nu$. Indeed, if n is large enough,

$$|a_n| < (1+\varepsilon)^n \qquad (^1)$$

and the ratio $t\frac{m+\nu}{p+\nu}$ of two consecutive coefficients decreases when ν increases. The sum of the terms of $\varphi_m(t)$ that follow $a_{m+\nu}t^\nu$ will therefore have a modulus smaller than

$$(1+\varepsilon)^{m+\nu}t^\nu\frac{(m+1)\dots(m+\nu)}{(p+1)\dots(p+\nu)}\left[1+t(1+\varepsilon)\frac{m+\nu+1}{p+\nu+1}+\left(t(1+\varepsilon)\frac{m+\nu+1}{p+\nu+1}\right)^2+\dots\right],$$

[1] ε represents here, and will represent in the following, a positive quantity that can be chosen as small as we wish, if n is large enough.

where $\lambda m \leq \nu < \lambda m + 1$. If ε is small enough and m large enough, the ratio of this geometric progression will differ as little as we wish from $t\frac{1+\lambda}{1+\lambda}$, which is smaller than 1. The mth root of the progression therefore tends towards 1, if m becomes infinite. On the other hand

$$\mathbf{L}\frac{(m+1)(m+2)\ldots(m+\nu)}{(p+1)(p+2)\ldots(p+\nu)} < \int_0^\nu \mathbf{L}\frac{m+x}{p+x}dx$$

$$= (m+\nu)\mathbf{L}\frac{m+\nu}{m} - (p+\nu)\mathbf{L}\frac{p+\nu}{p} + \nu\mathbf{L}\frac{m}{p},$$

$$\frac{1}{m}\mathbf{L}\left[(1+\varepsilon)^{m+\nu}t^\nu\frac{(m+1)\ldots(m+\nu)}{(p+1)\ldots(p+\nu)}\right]$$

$$< \frac{m+\nu}{m}\mathbf{L}\frac{(1+\varepsilon)(m+\nu)}{m} - \frac{p+\nu}{m}\mathbf{L}\frac{p+\nu}{p} + \frac{\nu}{m}\mathbf{L}\frac{mt}{p},$$

whose limit is

$$(1+\lambda)\mathbf{L}(1+\varepsilon) + (1+\lambda)\mathbf{L}(1+\lambda) - (t+\lambda)\mathbf{L}\left(1+\frac{\lambda}{t}\right).$$

This expression increases with t, and for $t = 1$ reduces to the first term, which is as small as we wish. Therefore, if $t < 1$, and if ε is small enough, it will be negative, and the mth root of the modulus of the sum of terms that follow $a_{m+\nu}$ has an upper limit, for $m = \infty$, smaller than 1.

Similarly, if $\lambda < t$, the sum of terms between a_{m-p} and $a_{m-\nu}$ has a modulus smaller than

$$(1+\varepsilon)^{m-\nu}\frac{1}{t^\nu}\frac{p(p-1)\ldots(p-\nu+1)}{m(m-1)\ldots(m-\nu+1)}$$

$$\times\left[1+\frac{p-\nu}{t(1+\varepsilon)(m-\nu)} + \left(\frac{p-\nu}{t(1+\varepsilon)(m-\nu)}\right)^2 + \ldots + \left(\frac{p-\nu}{t(1+\varepsilon)(m-\nu)}\right)^{p-\nu}\right];$$

the ratio of this progression will differ as little as we wish from $\frac{t-\lambda}{t(1-\lambda)}$, which is smaller than 1, and

$$\mathbf{L}\frac{p(p-1)\dots(p-\nu+1)}{m(m-1)\dots(m-\nu+1)} < \mathbf{L}\frac{p}{m} + \int_0^{\nu-1} \mathbf{L}\frac{p-x}{m-x}dx$$
$$= (m-\nu+1)\mathbf{L}\frac{m-\nu+1}{m} - (p-\nu+1)\mathbf{L}\frac{p-\nu+1}{p} + \nu\mathbf{L}\frac{p}{m},$$
$$\frac{1}{m}\mathbf{L}\left[(1+\varepsilon)^{m-\nu}\frac{1}{t^\nu}\frac{p(p-1)\dots(p-\nu+1)}{m(m-1)\dots(m-\nu+1)}\right] < \frac{m-\nu}{m}\mathbf{L}(1+\varepsilon)$$
$$+ \frac{m-\nu+1}{m}\mathbf{L}\frac{m-\nu+1}{m} - \frac{p-\nu+1}{m}\mathbf{L}\frac{p-\nu+1}{p} + \frac{\nu}{m}\mathbf{L}\frac{p}{mt},$$

whose limit

$$(1-\lambda)\mathbf{L}(1+\varepsilon) + (1-\lambda)\mathbf{L}(1-\lambda) - (t-\lambda)\mathbf{L}\left(1-\frac{\lambda}{t}\right)$$

is negative, if $t < 1$, and if ε is small enough.

Therefore, if in $\varphi_m(t)$ we keep only the terms between $a_{m-\nu}$ and $a_{m+\nu}$, the modulus of the sum of the suppressed terms will have an mth root inferior to a quantity smaller than 1, provided that m is large enough; and these terms will have no influence on the result stated, which leads to the following theorem:

Let

$$\varphi_m(t) = a_m + a_{m+1}t\frac{m+1}{p+1} + \dots + a_{m+\nu}t^\nu\frac{(m+1)\dots(m+\nu)}{(p+1)\dots(p+\nu)}$$
$$+a_{m-1}\frac{1}{t}\frac{p}{m} + \dots + a_{m-\nu}\frac{1}{t^\nu}\frac{p(p-1)\dots(p-\nu+1)}{m(m-1)\dots(m-\nu+1)},$$

where p is an integer varying with m in such a way that $\frac{p}{m}$ tends towards 1, and $\nu \geq \lambda m$, $0 < \lambda < t < 1$. If $z = 1$ is not a singular point of $f(z)$, the upper limit, for $m = \infty$, of $\sqrt[m]{|\varphi_m(t)|}$ is smaller than 1; and if this upper limit is 1, the point $z = 1$ is singular.

t is here an arbitrary quantity, which can vary with m, provided it remains between two fixed limits between 0 and 1, and that $\frac{p}{m}$ tends towards 1. We can, for example, suppose $t = \frac{p}{m}$.

Reciprocally *if, t remaining fixed, p varies in such a way that $m-p$ passes through all integer values, for example if p is between $mt-1$ and mt, and if $z = 1$ is a singular point, $\sqrt[m]{|\varphi_m(t)|}$ has upper limit 1; if this upper limit is smaller than 1, the point $z = 1$*

is not singular.

If we set $a_n = e^{2i(a'_n + ia''_n)}$, a' and a'' being real, $\varphi_m(t)$ takes the form $e^{2i(\varphi'_m + i\varphi''_m)}$, and the upper limit of $\sqrt[m]{|\varphi_m(t)|}$ is equal to the largest of the upper limits of $\frac{|\varphi'_m|}{m}$ and $\frac{|\varphi''_m|}{m}$. If, in $\varphi_m(t)$, we replace the terms a_n by the real part of $a_n e^{-2i\alpha}$, we see that, if the point $z = 1$ is not singular, $\sqrt[m]{|\varphi'_m|}$ will still have an upper limit smaller than 1, whatever α may be, which can thus vary with m. If $\sqrt[m]{|\varphi'_m|}$ has upper limit 1, the point $z = 1$ is singular.

3. Let $z = te^{i\theta}$; if $\sqrt[m]{|\varphi_m(te^{i\theta})|}$ has upper limit 1, the point $z = e^{i\theta}$ is singular. If, on an arc between $-\Omega$ and $+\Omega$, there is no singular point, this root $e^{i\theta m\omega}$ has an upper limit smaller than 1, provided that $-\Omega < \omega < +\Omega$, and the mth root of

$$\frac{1}{n}|e^{2i\alpha}\varphi_m(te^{i\omega_1}) + e^{2i\beta}\varphi_m(te^{i\omega_2}) + \ldots + e^{2i\gamma}\varphi_m(te^{i\omega_n})|$$

will also have an upper limit smaller than 1, whatever the arcs α. If this upper limit is 1, there will be at least one singular point on the arc $\pm\Omega$.

If the arcs ω_1, ω_2, ..., ω_n vary with m, in such a way that $|\omega|$ has, for $m = \infty$, an upper limit Ω, and if $\sqrt[m]{\frac{1}{n}\Sigma e^{2i\alpha}\varphi(te^{i\omega})}$ has upper limit 1, there will be a singular point on the arc $\pm(\Omega + \varepsilon)$. And if all the ω tend towards zero, there will be a singular point on the arc $\pm\varepsilon$, ε being as small as we wish; the point $z = 1$ will therefore be singular.

Let n positive arcs ω_1, ω_2, ..., ω_n, Ω their sum, and α_1, α_2, ..., α_n arbitrary arcs. We have

$$\begin{aligned}
&\cos(\alpha_1 + \nu\omega_1)\cos(\alpha_2 + \nu\omega_2)\ldots\cos(\alpha_n + \nu\omega_n) \\
&\quad = \frac{1}{2^n}\{e^{i(\alpha_1+\nu\omega_1)} + e^{-i(\alpha_1+\nu\omega_1)}\}\ldots\{e^{i(\alpha_n+\nu\omega_n)} + e^{-i(\alpha_n+\nu\omega_n)}\} \\
&\quad = \frac{1}{2^n}\Sigma e^{i[\pm(\alpha_1+\nu\omega_1)\pm(\alpha_2+\nu\omega_2)\pm\ldots\pm(\alpha_n+\nu\omega_n)]},
\end{aligned}$$

the sign Σ comprising the 2^n terms obtained by the permutation of the n signs $\pm$. Each of these terms is of the form

$$e^{i(\alpha+\nu\omega)}$$

where

$$|\omega| \leq \Omega.$$

In $\varphi_m(te^{i\omega})$, let us take for ω the 2^n values $\pm\omega_1 \pm \omega_2 \pm \ldots \pm \omega_n$, and for α the corresponding values. Then let

$$\begin{gathered}
\psi_m(t) = \frac{1}{2^n}\Sigma e^{i\alpha}\varphi_m(te^{i\omega}) \\
= a_m \cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_n + \ldots \\
+a_{m+\nu}e^{\nu}\frac{(m+1)\ldots(m+\nu)}{(p+1)\ldots(p+\nu)}\cos(\alpha_1+\nu\omega_1)\ldots\cos(\alpha_n+\nu\omega_n) \\
+a_{m-1}\frac{1}{t}\frac{p}{m}\cos(\alpha_1-\omega_1)\ldots\cos(\alpha_n-\omega_n)\ldots \\
+a_{m-\nu}\frac{1}{t^\nu}\frac{p(p-1)\ldots(p-\nu+1)}{m(m-1)\ldots(m-\nu+1)}\cos(\alpha_1-\nu\omega_1)\ldots\cos(\alpha_n-\nu\omega_n).
\end{gathered}$$

If $\omega_1+\omega_2+\ldots+\omega_n$ tends towards 0, and if $\sqrt[m]{|\psi_m(t)|}$ has upper limit 1, the point $z=1$ will be singular.

Suppose the terms a_n replaced by the real part a'_n of $a_n e^{-i\beta_n t}$, β_m being able to vary with m, and let us seek to determine the arcs α and ω in such a way that the terms of $\psi_m(t)$ are all of the same sign. If this can take place for an unlimited sequence of values of m such that $\sqrt[m]{|a'_m|}$ tends towards 1, $\omega_1+\omega_2+\ldots+\omega_n$ and $\frac{1}{m}\mathbf{L}|\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_n|$ tending towards 0, the point $z=1$ will be singular, because $\sqrt[m]{|a'_m\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_n|}$ will tend towards 1 for these values of m, and the other terms of $\psi_m(t)$ add to the first.

Let

$$\nu = \frac{\pi}{2} = \mu\omega_1,$$

μ being a positive or negative integer; then

$$\omega_n = \frac{\omega_1}{2^{n-1}}, \quad \alpha_n = -\mu\omega_n, \quad \text{where } n=2,3,\ldots$$

$\cos(\alpha_1 + \nu\omega_1)$ changes sign for the values

$$\nu = \mu + k' \frac{\pi}{\omega_1}$$

and $\cos(\alpha_n + \nu\omega_n)$ for

$$\nu = \mu + (2k+1)2^{n-2}\frac{\pi}{\omega_1} = \mu + k'\frac{\pi}{\omega_1}.$$

If $k' = 0$, $\cos(\alpha_1 + \nu\omega_1)$ is only null. If k' is divisible by 2^{n-2}, the quotient being even, $\cos(\alpha_n+\nu\omega_2)$ also cancels and the product $\cos(\alpha_1+\nu\omega_1)\cos(\alpha_2+\nu\omega_2)\ldots\cos(\alpha_n+\nu\omega_n)$ changes sign for $\nu = \mu$, and has no other sign change between the values $\nu = \mu \pm \frac{\pi}{\omega_1}2^{n-1}$. As in $\varphi_m(t)$ we can suppose $|\nu| \leq \lambda m$, it suffices to choose n such that $\frac{\pi}{\omega_1}2^{n-1}$ is larger than $\lambda m + |\mu|$. We then have

$$\omega_1 + \omega_2 + \ldots + \omega_n < 2\omega_1$$

and

$$\cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_n = \sin\mu\omega_1 \cos\frac{\mu\omega_1}{2}\cos\frac{\mu\omega_1}{2^2}\ldots\cos\frac{\mu\omega_1}{2^{n-1}}$$

$$= \frac{\sin\mu\omega_1}{2^{n-1}}\sum_{k=2^{n-1}}^{2^n-1} e^{\frac{i\mu\omega_1|2k+1|}{2^n}} = \frac{\sin^2\mu\omega_1}{2^{n-1}\sin\left(\frac{\mu\omega_1}{2^n}\right)},$$

and, if we suppose

$$0 < |\mu\omega_1| < \frac{\pi}{4},$$

we deduce

$$|\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_n| > \frac{\sin^2\mu\omega_1}{|\mu\omega_1|} \geq \frac{4}{\pi^2}|\mu\omega_1|.$$

a'_m being supposed positive, let $a'_{m+\nu_1}$, $a'_{m+\nu_2}$, ..., $a'_{m+\nu_q}$ be the first negative terms, then $a'_{m+\nu_{q+1}}$, $a'_{m+\nu_{q+2}}$, ..., the first following terms that are still positive, and so on, $a'_{m+\nu_1}$, $a'_{m+\nu_2}$, ..., $a'_{m+\nu_q}$ and $a'_{m-\nu_1}$, $a'_{m-\nu_2}$, ..., $a'_{m-\nu_q}$ being the terms that change sign between $a'_{m-\lambda m}$ and $a'_{m+\lambda m}$. Let us take successively for ν the values ν_1, ν_2, ..., ν_q, then $-\nu'_1$, ..., $-\nu'_q$, ω_1 keeping the same value $\omega_1 = \frac{\pi}{2\lambda m}$. Then $|\mu\omega|$ remains smaller than $\frac{\pi}{2}$; and if $\frac{q+q'}{m}$ tends towards 0, $\Sigma\omega < \pi\frac{q+q'}{\lambda m}$ will also tend towards 0.

On the other hand, we have

$$0 > \frac{1}{m}\Sigma\mathbf{L}|\cos\alpha| > \frac{1}{m}\Sigma\mathbf{L}\left|\frac{4}{\pi^2}\mu\omega_1\right| = \frac{q+q'}{m}\mathbf{L}\frac{2}{\pi\lambda m} + \frac{1}{m}\Sigma\mathbf{L}|\mu|$$

and

$$\Sigma\mathbf{L}|\nu_1| + \mathbf{L}(1.2\ldots q) + \mathbf{L}(1.2\ldots q') > \int_1^q \mathbf{L}x\,dx + \int_1^{q'} \mathbf{L}x\,dx$$
$$= q\mathbf{L}q + q'\mathbf{L}q' - q - q' + 2.$$

Therefore

$$\frac{1}{m}\Sigma\mathbf{L}|\cos\alpha| > \frac{q+q'}{m}\left(\mathbf{L}\frac{2}{\pi\lambda} - 1\right) - \frac{q}{m}\mathbf{L}\frac{m}{q} - \frac{q'}{m}\mathbf{L}\frac{m}{q'},$$

an expression that tends towards 0 at the same time as $\frac{q+q'}{m}$. We thus arrive at the following theorem:

β_m being a variable arc, let q be the number of sign changes of the real part a'_n of $a_n e^{-i\beta_n t}$, when n varies from $m - \lambda m$ to $m + \lambda m$. If, for an unlimited sequence of values of m, $\frac{1}{m}\mathbf{L}|a'_m|$ and $\frac{q}{m}$ tend towards 0, the point $z = 1$ is singular.

Suppose that between a'_m and $a'_{m+\lambda m}$ the number of sign changes is infinitely small compared to m; this theorem applies if $\frac{1}{n}\mathbf{L}|a'_n|$ tends towards 0 for a sequence of values of n between each $m(1+\lambda')$ and $m(1+\lambda-\lambda')$ where $0 < \lambda' < \lambda$; that is to say, if for all these values of n, $\frac{1}{n}\mathbf{L}|a'_n|$ has upper limit 0. If, on the contrary, this upper limit is smaller than 0, we can suppress $a'_n z^n$ without changing the singular points of the circle of convergence: so that the terms between $a_{m+\lambda' m}$ and $a_{m+\lambda-\lambda' m}$ reduce to the form $a''_n t e^{i\beta_n t}$.

4. We can further generalize this result by multiplying the series by a polynomial $P(z)$; if the point $z = 1$ is singular for the function $\mathbf{P}(z) \times f(z)$, it is singular for $f(z)$. The degree of this polynomial can even increase indefinitely. Suppose, indeed, that $z = 1$ is not singular; we could find a fixed quantity θ such that

$$\left|\frac{f^n(t)}{1.2\dots n}\right| < \left(\frac{1-\theta}{1-t}\right)^n,$$

provided that n is large enough. Let

$$\mathbf{P}(z) = \mathbf{A}_0 + \mathbf{A}_1 z + \mathbf{A}_2 z^2 + \dots + \mathbf{A}_\nu z^\nu,$$

$$\mathbf{Q}(z) = \mathbf{A}(1 + z + z^2 + \dots + z^\nu),$$

the coefficients $\mathbf{A}_0$, $\mathbf{A}_1$, ..., $\mathbf{A}_\nu$ having moduli smaller than $\mathbf{A}$, and ν being such that $n-\nu$ increases indefinitely with n. We have

$$\begin{aligned}
&\frac{1}{1.2\dots n}\frac{d^n\mathbf{P}(t)f(t)}{dt^n}\\
&= \frac{f^n(t)}{1.2\dots n}P(t) + \frac{f^{n-1}(t)}{1.2\dots(n-1)}\frac{\mathbf{P}'(t)}{1} + \dots + \frac{f^{(n-\nu)}(t)}{1.2\dots(n-\nu)}\frac{\mathbf{P}^{(\nu)}(t)}{1.2\dots\nu},\\
&\left|\frac{1}{1.2\dots n}\frac{d^n\mathbf{P}(t)f(t)}{dt^n}\right| < \left(\frac{1-\theta}{1-t}\right)^n\left[\mathbf{Q}(t) + \frac{1-\theta}{1-t}\mathbf{Q}'(t) + \dots + \left(\frac{1-\theta}{1-t}\right)^\nu\frac{Q^{(\nu)}(t)}{1.2\dots\nu}\right]\\
&= \left(\frac{1-\theta}{1-t}\right)^n\mathbf{Q}\left(t + \frac{1-\theta}{1-t}\right) < \left(\frac{1-\theta}{1-t}\right)^n\mathbf{A}\frac{\theta(1-t)}{(1-\theta)}\left(\frac{1-\theta t}{1-t}\right)^\nu < \mathbf{A}\frac{\theta(1-t)}{(1-\theta)}\left(\frac{1-\theta t}{1-t}\right)^\nu,
\end{aligned}$$

whose nth root has an upper limit, for $n=\infty$, smaller than $\frac{1}{1-t}$, if $\sqrt[n]{\mathbf{A}}$ has upper limit 1.

Let us set, as in No. 2, $n+p=m$, $\frac{p}{mt}$ tending towards 1, and form the function $\varphi_m(t)$, where each coefficient a_n is replaced by

$$b_n = \mathbf{A}_0 a_n + \mathbf{A}_1 a_{n-1} + \dots + \mathbf{A}_\nu a_{n-\nu},$$

$\frac{\mathbf{L}|\mathbf{A}|}{m}$ having upper limit 0 for $m=\infty$, and ν being such that $m(1-t)+\nu$ becomes infinite with m. The A can thus vary with m as well as ν. We can further neglect the non-included terms between $b_{m-\lambda m}$; and, if $\lambda < 1-t$, ν can be supposed equal to λm.

The same reasoning applies to all points of the circle of convergence, which allows us to perform the transformations of No. 3. Consider the sequence $b_m b_{m+1}\dots b_{m+\nu}$, where $\nu = \lambda m$. If we can find variable quantities A with m, $\frac{\mathbf{L}|\mathbf{A}|}{m}$ having upper limit

0 for $m=\infty$, such that, for an unlimited sequence of values of m, the real parts b'_m, b'_{m+1}, ..., $b'_{m+\nu}$ of the b have only a number of sign changes infinitely small compared to m, $\frac{\mathbf{L}|b'_{m+\mu}|}{m}$ tending towards 0 for values of μ between each $\lambda' m$ and $m(\lambda-\lambda')$; the point $z=1$ is singular.

Suppose that, for a sequence of values of m, the real parts of $a_n e^{-i\beta_n t}$ have, between $n=m$ and $m+\lambda m$, only a number of sign changes infinitely small compared to m. We have seen that the point $z=1$ is singular, except in the case where, by suppressing portions of terms such that $\frac{L|a'_n|}{n}$ has an upper limit smaller than 0, we obtain smaller intervals, but such that between a_m and $a_{m+\lambda' m}$ all terms are of the form

$$a''_n t e^{i\beta t}.$$

In this case, suppose the following terms put in the form

$$a_n = e^{it}(a'_n + i a''_n),$$

and suppress the common factor e^{it}. The quantities b' simplify if the A are supposed real. Because the a' are null for sequences of terms that can be represented by

$$a_{m-1} a_{m-2} \dots a_{m-\lambda m};$$

we then have

$$b'_m = A_0 a'_m, \quad b'_{m+1} = A_1 a'_{m+1} + A_1 a'_m, \quad \dots, \quad b'_{m+\nu} = A_\nu a'_{m+\nu} + \dots + A_0 a'_m,$$

where

$$\nu = \lambda m.$$

We can conclude that the point $z=1$ is singular when we can determine the quantities A such that the b' have only a number of sign changes infinitely small compared to m, $\frac{1}{m}L|b'_{m+\mu}|$ having upper limit 0 when μ takes values between $\lambda' m$ and $(\lambda-\lambda')m$.

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5. Let

$$a_n = \rho_n e^{i\omega_n t} = e^{it}(a'_n + ia''_n).$$

To determine the sign changes of $a'_n = \rho_n \cos(\omega_n - \beta)$, suppose we mark, on the circumference of radius 1, the arguments ω_n of the terms between a_m and $a_{m+\lambda m}$, by joining the consecutive points by the shortest arc, so as to form a continuous sinuous line. The sign of a'_n will change each time this line crosses one of the points $e^{(\beta+\frac{\pi}{2})t}$. The number q of sign changes of a'_n is equal to the number of intersection points of the preceding line with a diameter, which can be chosen to make q minimum. For each value of m, let β_m be the arc that corresponds to this minimum value of q, λ being able to vary, but without tending towards 0.

Suppose that $\frac{q}{m}$ has lower limit 0, for $m = \infty$, that is to say tends towards 0 for an unlimited sequence of values of m. We conclude that the point $z = 1$ is singular if, for the values of n between $m(1+\lambda')$ and $m(1+\lambda-\lambda')$, $\frac{1}{n}\mathbf{L}|\rho_n \cos(\omega_n - \beta_m)|$ has upper limit 0; that is to say if, for an unlimited sequence of these values of n, $\frac{1}{n}\mathbf{L}\rho_n$ and $\frac{1}{n}L|\cos(\omega_n - \beta_m)|$ tend towards 0.

This last expression tends towards 0 at the same time as $\frac{1}{n}\mathbf{L}|\omega_n - \beta_m - \frac{\pi}{2} \pm k\pi|$ (k being chosen so that the arc remains between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$), and in particular whenever

$$\omega_n - \beta_m = \frac{\pi}{2} \pm k\pi$$

does not tend towards 0.

For example, if, for each of these values of n, there are two arcs β and β' such that $\frac{q}{m}$ tends towards 0, $\frac{1}{n}L|\beta - \beta' \pm k\pi|$ tending towards 0, one of the expressions $\frac{1}{n}L|\omega_n - \beta - \frac{\pi}{2} \pm k'\pi|$, $\frac{1}{n}L|\omega_n - \beta' - \frac{\pi}{2} \pm k''\pi|$ will always tend towards 0. It only remains then to check if, for the values considered of n, $\frac{1}{n}L\rho_n$ has upper limit 0. But if this upper limit is smaller than 0, we can suppress these terms without changing the singular points of the circle of convergence.

Let a_{m-1}, a_{m-2}, ..., $a_{m-\lambda m}$ be one of the sequences of terms thus suppressed, m taking an unlimited sequence of values. Let $\nu = \lambda m$, and

$$b_m = \mathbf{A}_0 a_m, \quad b_{m+1} = \mathbf{A}_0 a_{m+1} + \mathbf{A}_1 a_m, \quad \ldots, \quad b_{m+\nu} = \mathbf{A}_0 a_{m+\nu} + \ldots + \mathbf{A}_\nu a_m,$$

the $\mathbf{A}$ being imaginary, variable with m, but such that $\frac{\mathbf{L}|\mathbf{A}|}{m}$ has upper limit 0. The point $z = 1$ will still be singular if we can choose these quantities $\mathbf{A}$ such that the b fill the conditions obtained for the a.

For example, if the $\mathbf{A}$ are all equal to e^{it}, we are led to construct the polygon whose vertices represent the quantities a_m, $a_m + a_{m+1}$, ..., $a_m + a_{m+1} + \ldots + a_{m+\nu}$. The real parts of b_m, b_{m+1}, ..., $b_{m+\nu}$ change sign each time this polygon crosses the diameter that corresponds to the directions $e^{-i(\beta \pm \frac{\pi}{2})t}$. We can choose the diameter that gives the number of intersection points q minimum; and, if $\frac{q}{m}$ tends towards 0, we conclude that the point $z = 1$ is singular, provided that, for the values of μ between $\lambda' m$ and $(\lambda - \lambda')m$, $\frac{1}{m}\mathbf{L}|b'_{m+\mu}|$ has upper limit 0.

6. It is easy to obtain fairly general cases where these methods will show that the point $z = 1$ is singular.

Form the differences $|\omega_{n+1} - \omega_n|$ between 0 and π, for the values of n between m and $m + \lambda m$. Among the diameters situated between two arcs β and $\beta + \gamma$, there will be at least one cutting the line that joins the arguments in a number q of points less than or at most equal to $\frac{1}{\gamma}\sum_{n=m}^{m+\lambda m} |\omega_{n+1} - \omega_n|$.

If $\frac{1}{m}\sum_{m}^{m+\lambda m} |\omega_{n+1} - \omega_n|$ tends towards 0, $\frac{q}{m}$ will tend towards 0 for several distinct diameters, and we can always choose β such that $\frac{1}{n}\mathbf{L}|\cos(\omega_n - \beta)|$ tends towards 0. Therefore, if $\frac{1}{n}\mathbf{L}|\rho_n|$ has upper limit 0, for the values of n between $m(1 + \lambda')$ and $m(1 + \lambda - \lambda')$, the point $z = 1$ is singular.

If, when m becomes infinite, $|\omega_{n+1}-\omega_n|$ tends towards 0 for the values of n between m and $m+\lambda m$, $\frac{1}{m}\sum_m^{m+\lambda m}|\omega_{n+1}-\omega_n|$ also tends towards 0. More generally, suppose that we divide the differences $|\omega_{n+1}-\omega_n|$ into two groups, the ones tending towards 0, the others being able to be arbitrary, $\frac{1}{m}\sum_m^{m+\lambda m}|\omega_{n+1}-\omega_n|$ will tend towards 0 if, in these intervals, the differences are all from the first group, except for a number q such that $\frac{q}{m}$ tends towards 0.

In particular, if $|\omega_{n+1}-\omega_n|$ tends regularly towards 0, for all infinite values of n, the point $z=1$ is singular, because we suppose that there always exist terms such that $\frac{L\rho_m}{m}$ tends towards 0.

The point $z=1$ is still singular whatever ρ_m, if $|\omega_{n+1}-\omega_n|$ tends towards 0 except for terms such that between a_m and $a_{m+\lambda m}$ there are only q, $\frac{q}{m}$ tending towards 0 for all infinite values of m.

Similarly, if $\omega_{n+1}-\omega_n$ tends towards a limit α for an infinite sequence of values of n, and if we can form an unlimited sequence of values of m such that between ω_m and $\omega_{m+\lambda m}$ there are only q differences that do not tend towards α, $\frac{q}{m}$ tending towards 0, $\frac{L\rho_n}{n}$ having upper limit 0, when n takes values between $m(1+\lambda')$ and $m(1+\lambda-\lambda')$, the point $z=e^{-i\alpha}$ is singular.

This can take place for several points, and even, in certain cases, for all points of the circle of convergence. Consider, for example, the series

$$\sum z^n\rho_n e^{i\omega_1\lambda x}$$

which gives

$$\omega_{n+1}-\omega_n=\sqrt{\mathbf{L}n+1}-\frac{n+1}{\sqrt{\mathbf{L}n+\sqrt{\mathbf{L}(n+1)}}}\left(1+\frac{1}{n}\right).$$

If n is large enough, $\omega_{n+1}-\omega_n$ will differ as little as we wish from $\sqrt{\mathbf{L}n}$.

On the other hand,

$$\sqrt{\mathbf{L}(n+p)} - \sqrt{\mathbf{L}n} = \frac{\mathbf{L}\left(1+\frac{p}{n}\right)}{\sqrt{\mathbf{L}n + \sqrt{\mathbf{L}(n+p)}}},$$

which tends towards 0 if $\frac{p}{n}$ remains finite, or even if $\frac{\mathbf{L}\frac{p}{n}}{\sqrt{\mathbf{L}n}}$ tends towards 0.

Let α be an arc between 0 and 2π. Consider the sequence of values of n between $e^{(\alpha+2k\pi)^2-2\pi\alpha}$ and $e^{(\alpha+2k\pi)^2+2\pi\alpha}$, t tending towards 0 when k becomes infinite. $\omega_{n+1}-\omega_n$ tends towards α for all the values of n between n and $n+\lambda n$, and the point $e^{-i\alpha}$ is singular whatever α, provided that $\frac{1}{n}\mathbf{L}\rho_n$ has upper limit 0 for the values of n that correspond to each arc α. This takes place, in particular, if $\frac{\mathbf{L}\rho_n}{n_0}$ tends regularly towards 0 for a sequence n_1, n_2, ..., n_ν of values of n such that $\frac{n_{\nu+1}}{n_\nu}$ remains finite, or even more generally such that $\frac{1}{\sqrt{\mathbf{L}n_\nu}}\mathbf{L}\frac{n_{\nu+1}}{n_\nu}$ tends towards 0.

We arrive at the same result for series of the form:

$$\sum z^n \rho_n e^{in\varphi(n)},$$

$\varphi(n)$ being such that $n|\varphi(n+1)-\varphi(n)|$ tends towards 0, $\varphi(n)-2k\pi$ varying constantly between 0 and 2π when n increases indefinitely; which takes place in particular if $\varphi(n)$ increases indefinitely, $n\varphi'(n)$ tending towards 0.

Similarly in the series

$$\sum z^n \rho_n e^{in\alpha+in\varphi(n)},$$

$\omega_{n+1}-\omega_n$ has for limit the values between $-\alpha$ and $+\alpha$; and if $\alpha<\pi$, this reasoning only applies to a portion of the circle of convergence.

7. If the series contains a sequence of terms such that $\frac{\mathbf{L}|a_n|}{n}$ has, for $n=\infty$, an upper limit smaller than 0, we can suppress them, and there is no longer need to take account of their signs. Suppose that, for a sequence of values of m such that $\frac{L|a_m|}{m}$ tends towards 0, there remain between $a_{m\pm\lambda m}$ only q terms, $\frac{q}{m}$ tending towards 0. By

making the substitution $z = z'e^{i\omega}$, the real parts of the coefficients of the new series will have at most q sign changes in the same interval, and the point $z' = 1$, $z = e^{i\omega}$ will be singular whatever ω.

If, for a sequence of values of m, there remain between a_m and $a_{m+\lambda m}$ only q terms, $\frac{q}{m}$ tending towards 0, and if $\frac{\mathbf{L}|a_n|}{n}$, where n remains between $m(1+\lambda')$ and $m(1+\lambda-\lambda')$, has upper limit 0, all points of the circle of convergence are singular.

This takes place, in particular, for the series

$$\sum c_\nu z^\nu,$$

where $c_{\nu+1} - c_\nu$ increases indefinitely with ν, and even if, ρ being a fixed number, $c_{\nu+\rho} - c_\nu$ increases indefinitely, and indeed in the more general case where $c_{\nu+1} - c_\nu$ increases indefinitely, except for terms such that, between c_ν and $(1+\lambda)c_\nu$, there are only an infinitely small number compared to c_ν, when ν increases indefinitely. All points of the circle of convergence are then singular whatever the coefficients a.

We can further multiply the series by a polynomial with variable coefficients, and replace a_n by

$$b_n = \mathbf{A}_0 a_n + \mathbf{A}_1 a_{n-1} + \ldots + \mathbf{A}_\nu a_{n-\nu},$$

the coefficients $\mathbf{A}$ satisfying the conditions indicated in No. 4.

In particular, suppose that, for an unlimited sequence of values of m, the terms a_n between a_m and $a_{m-\lambda m}$ are such that $\frac{\mathbf{L}|a_n|}{n}$ has an upper limit smaller than 0. We can then suppress these terms and we will have

$$b_m = \mathbf{A}_0 a_m,$$

$$b_{m+1} = \mathbf{A}_0 a_{m+1} + \mathbf{A}_1 a_m,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$b_{m+\nu} = \mathbf{A}_0 a_{m+\nu} + \ldots + \mathbf{A}_\nu a_m,$$

where $\nu = \lambda m$. We will still be sure that all points of the circle of convergence are singular if we can choose the variable quantities $\mathbf{A}$ with m, such that $\frac{\mathbf{L}|\mathbf{A}|}{m}$ has upper limit 0, the b being null, except for an infinitely small number compared to m, $\frac{\mathbf{L}|b_n|}{n}$ having upper limit 0 when n remains between m and $m(1+\lambda-\lambda')$.

Let $z = z'e^{i\omega t}$ and $\mathbf{A}_\nu = \mathbf{A}'_\nu e^{i\nu\omega t}$. We can recognize that the point $z = e^{i\omega t}$ is singular if in the sequence

$$b_m = \mathbf{A}'_0 a_m,$$

$$b_{m+1} = e^{i\omega t}(\mathbf{A}'_0 a_{m+1} + \mathbf{A}'_1 a_m),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$b_{m+\nu} = e^{i\nu\omega t}(\mathbf{A}'_0 a_{m+\nu} + \ldots + \mathbf{A}'_\nu a_m)$$

the real parts of the b have only a number of sign changes infinitely small compared to m, the mth root of the real part of b_n having upper limit 1, when n remains between m and $m(1+\lambda-\lambda')$. And all points of the circle of convergence will be singular if these conditions can be satisfied whatever ω, the A being able to vary with ω.

This is what takes place, for example, if, for an unlimited sequence of values of m, $a_{m-1} = a_{m-2} = \ldots = a_{m-\lambda m} = 0$, $a_m a_{m+1} \ldots a_{m+\lambda m}$ being real, positive and decreasing, $\sqrt[m]{a_m}$ tending towards 1. Indeed, if we take $\mathbf{A}_0 = \mathbf{A}_1 = \ldots = \mathbf{A}_\nu = -ie^{\frac{\omega t}{2}}$ and $z = z'e^{i\omega t}$, we will have

$$b_{m+\nu} = -ie^{\frac{\omega t}{2}}(a_m + a_{m+1}e^{i\omega t} + \ldots + a_{m+\nu}e^{i\nu\omega t})$$

and the real part of $b_{m+\nu}$ is equal to

$$a_m \sin\frac{\omega}{2} + a_{m+1}\sin\frac{3\omega}{2} + \ldots + a_{m+\nu}\sin\frac{2\nu+1}{2}\omega$$

$$= \frac{1}{\sin\frac{\omega}{2}}\left[(a_m - a_{m+1})\sin^2\frac{\omega}{2} + (a_{m+1} - a_{m+2})\sin^2\frac{3\omega}{2} + \ldots\right.$$

$$\left. + (a_{m+\nu-1} - a_{m+\nu})\sin^2\frac{2\nu\omega}{2} + a_{m+\nu}\sin^2\frac{2\nu+1}{2}\omega\right],$$

an expression whose terms all have the sign of $\sin\frac{\omega}{2}$, and $\sqrt[m]{|a_m \sin\frac{\omega}{2}|}$ will tend towards 1, provided that $\sin\frac{\omega}{2}$ is not null.

We thus arrive at the same result in the more general case where we can determine the quantities $\mathbf{A}$ such that the b are real, positive and decreasing, $\sqrt[m]{b_m}$ tending towards 1, b_{m-1}, b_{m-2}, ..., $b_{m-\lambda m}$ being null; because we can multiply successively by two polynomials.

8. To obtain new applications of the established theorems in §2 and 3, we will now calculate the order of magnitude of the coefficients of $\varphi_m(t)$ when m becomes infinite, supposing $t = \frac{p}{m}$, p being an integer that varies with m, such that t remains between two fixed limits between 0 and 1. We then have

$$\mathbf{L}t^\nu \frac{(m+1)\dots(m+\nu)}{(p+1)\dots(p+\nu)} = \sum_{x=1}^{\nu} \mathbf{L}\frac{1+\frac{x}{m}}{1+\frac{x}{p}}$$

$$= -\frac{1-t}{mt}\sum_1^\nu x + \frac{1-t^2}{2m^2t^2}\sum_1^\nu x^2 - \frac{1-t^3}{3m^3t^3}\sum_1^\nu x^3 + \dots;$$

or

$$\frac{1-t^n}{n} < \frac{1-t^{2-1}}{n-1} \quad \text{and} \quad \sum_1^\nu x^n < \nu\sum_1^\nu x^{n-1}.$$

Consequently, the ratio of two consecutive terms

$$\frac{1-t^n}{1-t^{n-1}}\frac{n-1}{nt}\frac{\Sigma x^n}{\Sigma x^{n-1}}$$

is smaller than $\frac{\nu}{mt} < \frac{\lambda}{t} < 1$ if $\lambda < t$. We thus have

$$Lt^\nu \frac{(m+1)\dots(m+\nu)}{(p+1)\dots(p+\nu)} < -\frac{1-t}{2mt}\nu(\nu+1)$$

$$+\frac{1-t^2}{2m^2t^2}\frac{\nu(\nu+1)(2\nu+1)}{6} < -\frac{1-t}{2mt}\nu(\nu+1)(1-\alpha),$$

α being able to be as small as we wish if λ is small enough.

Similarly

$$\mathbf{L}\frac{1}{p}\frac{p(p-1)\dots(p-\nu+1)}{m(m-1)\dots(m-\nu+1)} = \sum_{x=1}^{\nu-1}\mathbf{L}\frac{1-\frac{x}{p}}{1-\frac{x}{m}} = \frac{1-t}{mt}\sum_{1}^{\nu-1}x - \frac{1-t^2}{m^2t^2}\sum_{1}^{\nu-1}x^2 + \dots$$

$$< -\frac{1-t}{2mt}\nu(\nu-1) < -\frac{1-t}{2mt}\nu(\nu+1)(1-\alpha),$$

provided that $\nu > \frac{2-\alpha}{\alpha}$.

Let $\frac{1-t}{2t}(1-\alpha) = k$. The coefficient of $a_{m+\nu}$ in $\varphi_m(t)$ is smaller than $e^{-k\frac{\nu(\nu+1)}{m}}$, and that of $a_{m-\nu}$ is smaller than $e^{-k\frac{1-t}{2mt}\nu(\nu-1)}$, and smaller than $e^{-k\frac{\nu(\nu+1)}{m}}$ if $\nu > \frac{2-\alpha}{\alpha}$.

The sum of coefficients that follow that of $a_{m+\nu}$ is smaller than

$$\sum_{\nu'=\nu+1}^{\lambda m} e^{-k\frac{\nu'(\nu'+1)}{m}} < \int_{\nu}^{\infty} e^{-k\frac{x(x+1)}{m}}dx = e^{-k\frac{\nu(\nu+1)}{m}}\int_{0}^{\infty} e^{-k\frac{x(x+2\nu+1)}{m}}dx$$

$$< e^{-k\frac{\nu(\nu+1)}{m}}\int_{0}^{\infty} e^{-k\frac{x(2x+1)}{m}}dx = \frac{m}{k(2\nu+1)}e^{-k\frac{\nu(\nu+1)}{m}}.$$

We also have

$$\int_{\nu}^{\infty} e^{-k\frac{x(x+1)}{m}}dx < e^{-k\frac{\nu(\nu+1)}{m}}\int_{0}^{\infty} e^{-k\frac{x^2}{m}}dx = \frac{1}{2}\sqrt{\frac{\pi m}{k}}e^{-k\frac{\nu(\nu+1)}{m}};$$

an expression that is smaller than the preceding if $\nu < \sqrt{\frac{m}{k\pi}} - \frac{1}{2}$.

The sum of coefficients situated beyond that of $a_{m-\nu}$ is smaller than

$$\sum_{x=\nu+1}^{\lambda m} e^{-\frac{1-t}{2mt}x(x-1)} < \int_{\nu}^{\infty} e^{-\frac{1-t}{2mt}x(x-1)}dx < \int_{\nu}^{\infty} e^{-k\frac{x(x-1)}{m}}dx,$$

if $\nu \geq \frac{2-\alpha}{2}$. But if $0 \leq \nu < \frac{2-\alpha}{2}$, the difference of the last two integrals is always smaller than

$$\int_0^\alpha e^{-\frac{1-t}{2mt}x(x-1)}dx - \int_0^\alpha e^{-\frac{1-t}{2m}(1-\alpha)(x-1)}dx$$

$$= \int_0^1 e^{-\frac{1-t}{2mt}x(x-1)}dx + \int_0^1 e^{-\frac{1-t}{2m}(1-\alpha)x(x-1)}dx + \int_0^\alpha e^{-\frac{1-t}{2mt}(x^2-1)}dx$$

$$= \int_0^\alpha e^{-\frac{1-t}{2mt}[1-\alpha(2\nu-1)]}dx < e^{\frac{1-t}{2mt}}\left[1+\sqrt{\frac{\pi mt}{2(1-t)}} - e^{-\frac{1-t}{2}\alpha}\sqrt{\frac{\pi mt}{2(1-t)(1-\alpha)}}\right],$$

an expression that remains negative if m is large enough.

Therefore the sum of coefficients coming after that of $a_{m-\nu}$, similarly that the sum of those that follow that of $a_{m+\nu}$, is smaller than the smaller of the expressions $\frac{m}{k(2\nu+1)}e^{-k\frac{\nu(\nu+1)}{m}}$, $\frac{1}{2}\sqrt{\frac{\pi m}{k}}e^{-k\frac{\nu(\nu+1)}{m}}$. And the sum of all coefficients is smaller than $\sqrt{\frac{\pi m}{k}}$.

In $\varphi_m(t)$ suppose the quantities a_n replaced by the real part a'_n of $a_n e^{-i\beta t}$. If we consider terms such that $|a'| < \mathbf{A}$, their sum will be smaller than $A\sqrt{\frac{m\pi}{k}}$. If these terms are not between $a'_{m+\nu}$, the sum will be smaller than $\mathbf{A}\frac{\lambda m}{k(2\nu+1)}e^{-k\frac{\nu(\nu+1)}{m}}$.

9. Let, for an unlimited sequence of values of m, $\mathbf{A}_m$ be a positive quantity, variable with m, such that $\frac{\mathbf{L}\mathbf{A}_m}{m}$ tends towards 0. Suppose that there exist in $\varphi_m(t)$ terms such that $|a'_{m\pm\nu}|\sqrt[m]{\frac{m}{\mathbf{A}_m}}$ remains smaller than a given quantity. Let ν be a variable number with m, that we can suppose larger than $\sqrt{m}$; suppose there exist terms, not between $a'_{m\mp\nu}$, such that $\frac{m}{2}\mathbf{L}\left|\frac{a'}{\mathbf{A}_m} < \frac{m}{2}\right|$ remains equally finite. In the case where $\frac{\nu}{\sqrt{m\mathbf{L}m}}$ does not tend towards 0, it suffices even that $\frac{m}{2}\mathbf{L}\left|\frac{a'}{\mathbf{A}_m}\right|$ remains finite. We can always choose k large enough, or t small enough, so that the sum of all these terms in $\varphi_m(t)$ is smaller than A_m multiplied by a fixed quantity θ as small as we wish.

Let ε_m be the largest of the quantities $\frac{1}{m}\mathbf{L}\left|\frac{a'_{m\pm\nu}}{\mathbf{A}_m}\right|$, where $\nu \leq \lambda m$; when m becomes infinite, ε_m tends towards 0. If $\varepsilon_m < \frac{\mathbf{L}m}{m}$, suppose $\nu > \sqrt{\mathbf{H}m\mathbf{L}m}$, we then have

$$\frac{m}{\nu^2}\mathbf{L}\left|\frac{a'}{\mathbf{A}_m}\frac{m}{\nu}\right| < \frac{3}{\lambda II}.$$

If $\varepsilon_m > \frac{\mathbf{L}m}{m}$ and $\nu > m\sqrt{\mathbf{H}\varepsilon_m}$, we will have

$$\frac{m}{\nu^2}\mathbf{L}\left|\frac{a'}{\mathbf{A}_m}\frac{m}{\nu}\right| < \frac{1}{m\mathbf{H}\varepsilon_m}\left[m\varepsilon_m + \frac{1}{\lambda}L\frac{1}{\mathbf{H}\varepsilon_m}\right] < \frac{3}{\lambda\mathbf{H}}.$$

Therefore, $\mathbf{H}$ being a fixed quantity that can be very small, the group of terms that we are considering will comprise all those for which ν is greater than the larger of the two quantities $m\sqrt{\mathbf{H}\varepsilon_m}$, $\sqrt{\mathbf{H}m\mathbf{L}m}$. We can still, after having removed terms such that $\left|\frac{a'}{\mathbf{A}_m}\sqrt{m}\right|$ or $\frac{m}{\nu^2}\mathbf{L}\left|\frac{a'}{\mathbf{A}_m}\times\frac{m}{\nu}\right|$ remains finite, search to make the others of the same sign by the method of n° 3. Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be the absolute values of the numbers $\pm\nu_1$ that correspond to sign changes of the terms $a'_{m+\nu}$ or $a'_{m-\nu}$ that we have kept between $a_{m\pm\lambda m}$. Supposing that $\frac{q}{m}$ tends towards 0, we have seen that we can form a function $\psi_m(t)$ having all its terms positive, the first a'_m being multiplied by the factor

$$\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_q > \left(\frac{2}{\pi\lambda m}\right)^q\mu_1\mu_2\ldots\mu_q;$$

or

$$L\mu_1 + L\mu_2 + \ldots + L\mu_q > \sum_{\nu=1}^{q} L\frac{\nu}{3} > \int_1^q L\frac{x}{3}dx = qL\frac{q}{3} - q + 2$$

and

$$\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_q > \left(\frac{q}{\pi e\lambda m}\right)^q.$$

But we can obtain a function $\psi_m(t)$ for which this first term will have a larger value. Let ε be a quantity that tends towards 0, but also slowly enough that we want, for example $\varepsilon = \frac{1}{\mathbf{L}m}$ or $\frac{1}{\mathbf{L}(\mathbf{L}m)}$. The numbers $\mu_1, \mu_2, \ldots, \mu_q$ being arranged in order of increasing magnitude, let us form the sequence $\frac{q}{q}, \frac{q-1}{q-1}, \ldots, \frac{1}{1}$. Let h be the number of these quantities greater than ε, and $\frac{1}{\mu_q}$ the first that is smaller than ε, so that $\nu > h$. For the $q-h$ values of μ such that $\frac{\mu}{\mu_q} < \varepsilon$, let us make correspond the arc $\omega_1 = \frac{\pi}{4q} \times \frac{\mu}{\mu_q}$; then $\mu\omega$ remains smaller than $\frac{\pi}{4}$, and $\Sigma\omega < (q-h)\frac{\pi}{q}\varepsilon < \pi\varepsilon$. For the h other values of μ, let us make correspond the arc $\omega_1 = \frac{\pi}{4}\varepsilon$, which gives

$$\mu_n\omega_1 = \frac{\pi}{4}\varepsilon\mu_n < \frac{\pi}{4}\frac{\mu_n}{\mu_q} < \frac{\pi}{3} \quad \text{and} \quad \Sigma\omega < \frac{h}{q}\pi\varepsilon < \pi\varepsilon.$$

Thus $\Sigma\omega$ tends towards 0, and $\mu\omega_1$ remains smaller than $\frac{\pi}{3}$. We then have

$$\sum \mathbf{L}\cos\alpha > \sum^{h} \mathbf{L}\left(\frac{2}{\pi}\frac{h}{q}\right) + \sum^{q-h} \mathbf{L}\left(\frac{2}{\pi}\varepsilon\frac{\mu_q}{q}\right)$$

$$= q\mathbf{L}\frac{2}{\pi} - (q-h)\mathbf{L}q + h\mathbf{L}\frac{h}{q} + \sum^{h} \mathbf{L}n + \sum^{q-h} L\mu_n.$$

The sign $\sum^h$ applying to the $q-h$ values of n such that $\frac{n}{\mu_n} < \varepsilon$, the sign $\sum^{q-h}$ to the h values of n such that $\frac{n}{\mu_n} > \varepsilon$. But $\mu_n < \frac{n}{q}$ and

$$\sum^{h} \mathbf{L}n + \sum^{q-h} \mathbf{L}\mu_n > \sum_{n=1}^{q} \mathbf{L}n = h\mathbf{L}q - q + h\mathbf{L}2,$$

$$\sum \mathbf{L}\cos\alpha > -q\left(1 + \mathbf{L}\frac{\pi}{2}\right) + h\mathbf{L}\frac{q^2}{2}$$

and

$$\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_q > \left(\frac{2}{\pi e}\right)^q \left(\frac{\varepsilon}{2}\right)^h > \left(\frac{\varepsilon}{\pi e}\right)^q.$$

a'_m being supposed positive, $\psi'_m(t)$ will be larger than

$$a'_m\left(\frac{\varepsilon}{2}\right)^q\left(\frac{2}{\pi e}\right)^q - \theta\mathbf{A}_m,$$

and we will be sure that the point $z = 1$ is singular if we can determine ε tending towards 0, such that $\frac{a'_m}{A_m}\left(\frac{\varepsilon}{2}\right)^q\left(\frac{2}{\pi e}\right)^q$ does not tend towards 0, that is to say if $\frac{1}{q}\left(\mathbf{L}\frac{a'_m}{A_m} - q\mathbf{L}\frac{\pi e}{2}\right)$ increases indefinitely, through positive values. This happens, in particular, if $\frac{1}{q}\mathbf{L}\frac{a'}{\mathbf{A}_m}$ increases indefinitely.

We can still apply here the principles of n° 5. We will first remove terms such that $\left|\frac{a}{\mathbf{A}_m}\sqrt{m}\right|$ or $\frac{m}{\nu^2}\mathbf{L}\left|\frac{a}{\mathbf{A}_m}\times\frac{m}{\nu}\right|$ remains finite. For each value of m, we will determine the arc β such that q is minimum. If $\frac{1}{q}\mathbf{L}\left|\frac{a_m}{\mathbf{A}_m}\cos(\omega_m - \beta)\right|$ increases indefinitely, for an unlimited sequence of values of m, the point $z = e^{i\omega}$ is singular. In the case where $\frac{h}{q}$ tends towards 0, the point $z = 1$ is still singular if $\frac{1}{q}\mathbf{L}\left|\frac{a_m}{\mathbf{A}_m}\cos(\omega_m - \beta)\right|$ remains larger than a quantity greater than $\mathbf{L}\frac{\pi e}{2}$.

10. If, after having removed terms such that $\left|\frac{a}{A_m}\sqrt{m}\right|$ or $\frac{m}{\nu^2}L\left|\frac{a}{A_m}\times\frac{m}{\nu}\right|$ remains finite, there remain in $\varphi_m(t)$ only q terms, $\frac{q}{m}$ tending towards 0, making the substitution $z = e^{i\omega e^{it}}$, the real parts of the terms kept in the new series will have at most q sign changes; and, if $\frac{1}{q}L\left|\frac{a_m}{A_m}\right|$ increases indefinitely, the point $z = e^{i\omega}$ will be singular whatever ω may be.

We arrive at even more general results by noting that, to form here the function $\psi_m(t)$ of n° 3, it suffices to make correspond to each value of μ a single arc ω, and the arc $\alpha = \frac{\pi}{2} \mp \mu\omega$, such that $\cos(\alpha \mp \mu\omega) = 0$. $a_{m\mp\mu}$ representing the terms kept in $\varphi_m(t)$, suppose the μ divided into three groups: the first such that $\sum\frac{1}{\mu}$ tends towards 0; to these values we will make correspond the arc $\omega = \frac{\pi}{2\mu}$, so that $\cos\alpha = 1$, $\Sigma\omega$ tending towards 0. $\mu_1, \mu_2, \ldots, \mu'_q$ being the other values arranged in order of magnitude, the second group will comprise the $q' - h$ values of μ such that $\frac{n}{\mu_n} < \varepsilon$, and the third the h values such that $\frac{n}{\mu_n} > \varepsilon$. ε is still a quantity that tends towards 0 when m becomes infinite. For the terms of the third group we take

$$\omega = \frac{\pi}{3} \cdot \frac{\varepsilon}{q'}$$

and for the second

$$\omega = \frac{\pi}{3q'} \times \frac{n}{\mu_n}.$$

Then $\mu\omega$ remains smaller than $\frac{\pi}{3}$ and $\sum \omega < \frac{\pi}{3}\varepsilon$ tends towards 0. We still have

$$\sum \mathbf{L} \cos\alpha = \sum{}' \mathbf{L} \sin\frac{\pi n}{3q'} + \sum{}'' \mathbf{L} \sin\frac{\pi\varepsilon\mu_n}{3q'} > \sum{}' \mathbf{L}\frac{n}{q'} + \sum{}'' \mathbf{L}\frac{\varepsilon\mu_n}{q'} > h\mathbf{L}\frac{\varepsilon}{3} - q',$$

$$\cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_{q'} > e^{-q'} \left(\frac{\varepsilon}{3}\right)^h > \left(\frac{\varepsilon}{3e}\right)^{q'}.$$

The function $\psi_m(t)$ then reduces to the single term $a_m \cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_{q'}$ and the neglected terms have a sum smaller than θA_m.

If $\frac{1}{h}\left(\mathbf{L}\left|\frac{a_m}{A_m}\right| - q'\right)$ increases indefinitely, we can take

$$\mathbf{L}\frac{\varepsilon}{3} = \frac{1}{h}\left(\mathbf{L}q' + q' - \mathbf{L}\left|\frac{a_m}{A_m}\right|\right).$$

Then ε tends towards 0 and we can suppose

$$h < q' = \left|\frac{a_m}{\mathbf{A}_m}\right| \left(\frac{\varepsilon}{3}\right)^h \times e^{-q'} \quad \text{and} \quad |\psi_m(t)| > \mathbf{A}_m(q' - \theta).$$

All points of the circle of convergence are then singular. This happens, in particular, if $\frac{1}{q'}L\left|\frac{a_m}{A_m}\right|$ increases indefinitely; or if, $\frac{h'}{q'}$ tending towards 0, $\frac{1}{q'}L\left|\frac{a_m}{A_m}\right|$ remains greater than a quantity larger than 1.

All points of the circle of convergence are still singular if $h = 0$, $L\left|\frac{a_m}{A_m}\right| - q'$ remaining greater than a given quantity that can be negative; and finally, in the case where $q' = 0$, $\left|\frac{a_m}{A_m}\right|$ remaining greater than a given quantity larger than 0.

We can note that $q' = 0$ in the particular case where the terms that we keep reduce to the form $\sum a_\nu z^{c_\nu}$, $\frac{c_{\nu+1} - c_\nu}{\mathbf{L}c_\nu}$ increasing indefinitely with c_ν. For if $m = c_\nu$, a_{c_ν} being any term of $\varphi_m(t)$, $\frac{c_{\nu+1} - c_\nu}{\mathbf{L}c_\nu}$ also increases indefinitely, and we could find a quantity $\mathbf{A}$ that increases indefinitely with ν, such that, for these terms, $|c_{m-p} - c_\nu| > \mathbf{A}p\mathbf{L}c_\nu$; and as $|c_{m-p} - c_\nu| < \lambda c_\nu$, we will have

$$p < \frac{\lambda c_\nu}{\mathbf{AL}c_\nu};$$

consequently,

$$\sum \frac{1}{\mu} < \frac{2}{\mathbf{AL}c_\nu}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p}\right)$$
$$< \frac{2}{\mathbf{AL}c_\nu}(1 + \mathbf{L}p) < \frac{2}{\mathbf{A}}\left[1 + \frac{1 + \mathbf{L}\lambda - \mathbf{L}(\mathbf{AL}c_\nu)}{Lc_\nu}\right],$$

an expression that tends towards 0 if $\mathbf{A}$ increases indefinitely or even at the same time as c_ν.

This happens, for example, if $\frac{c_\nu}{\nu \mathbf{L}\nu}$ increases constantly and indefinitely with ν; or again if $\frac{c_\nu}{\mathbf{L}\nu}$ grows constantly, $\frac{c_\nu}{\nu L \nu^2}$ increasing indefinitely.

Consider, for example, the series

$$\sum z^n \cdot e^{\omega_n i} \cdot e^{n^2 \cos(2\pi n^2)},$$

where $0 < \beta < 1$, γ being a commensurable number smaller than 1 and greater than the two quantities $1 - \frac{\beta}{2}$, $\frac{3+\beta}{4}$.

Let us form $\varphi_m(t)$ for values of m such that m^2 is an integer, and let

$$\mathbf{A}_m = |a_m| = e^{m^2};$$

$\sqrt[m]{|a_m|}$ then tends towards 1. N being any integer, let us divide the terms a into two groups, the first comprising the terms a_n such that

$$\mathbf{N}^{\frac{1}{\gamma}} - \frac{1}{2} < n < \mathbf{N}^{\frac{1}{\gamma}} + \frac{1}{2}.$$

For the terms of the second group we will have

$$\mathbf{N}^{\frac{1}{\gamma}} + \frac{1}{2} < n < (\mathbf{N}+1)^{\frac{1}{\gamma}} - \frac{1}{2}$$

and if $\mathbf{N}$ is large enough, provided that $\gamma' < \gamma$, we will have

$$\mathbf{N} + \frac{\gamma'}{2} n^{\gamma-1} < n^{\gamma} < \mathbf{N} + 1 - \frac{\gamma'}{2} n^{\gamma-1},$$

$$\cos(2\pi n^{\gamma}) < 1 - 2\gamma'^2 n^{-2(1-\gamma)};$$

so that for these terms

$$\left|\frac{a_{m\mp\nu}}{\mathbf{A}_m}\right| < e^{(m\mp\nu)^2 + 2\gamma'^2 (m\mp\nu)^{1+\gamma(1-\gamma)} - m^2} < e^{[\mp 2m\nu^2 - 2\gamma'^2 m^{2-\gamma(1-\gamma)}]},$$

and $\left|\frac{a_{m\mp\nu}}{\mathbf{A}_m}\sqrt{m}\right|$ tends towards 0 if $\frac{\nu}{m^{2\gamma-1}}$ remains smaller than a quantity smaller than $\frac{2\gamma^2}{\beta}$.

On the other hand, in $\varphi_m(t)$,

$$|a| < e^{m^2(1+\lambda)^2} \quad \text{and} \quad \frac{m}{\nu^2}\mathbf{L}\left|\frac{a}{\mathbf{A}_m}\cdot\frac{m}{\nu}\right|$$

will remain finite if $\frac{m^{1+\beta}}{\nu^2}$ remains finite. And if $\gamma > \frac{3+\beta}{4}$, we should only keep in $\varphi_m(t)$ the terms of the first group, which are of the form $a_N z^{c_{\mathbf{N}}}$, where $\frac{c_{\mathbf{N}}}{\mathbf{NLN}}$ increases constantly and indefinitely with $\mathbf{N}$.

Consequently, all points of the circle of convergence are singular.

11. To apply the preceding results, we are led to study the order of magnitude of the coefficients a_m of the series, when m becomes infinite. Let us first consider the terms of modulus larger than 1; $\frac{\mathbf{L}|a_m|}{m}$ tending towards 0, we can set $\frac{\mathbf{L}|a_m|}{m} = \frac{1}{m/\alpha}$; α remaining positive if m is large enough. $\frac{|\mathbf{L}|a_m||}{Lm} = 1 - \alpha$ remains smaller than 1, but can nevertheless tend towards 1, in the case where α tends towards 0, such that m^{α} becomes infinite, for example, if $|a_m| = e^{\frac{m}{\mathbf{L}m}}$; so that, for terms of modulus larger

than 1, $\frac{\mathbf{LL}|a_m|}{\mathbf{L}m}$ has an upper limit, for $m = \infty$, comprised between 1 and $-\infty$.

Let β be this upper limit, which we will suppose finite. We could find an unlimited sequence of values of m such that

$$e^{m^{\beta+\varepsilon}} > |a_m| > e^{m^{\beta-\varepsilon}};$$

$\frac{\mathbf{L}|a_m|}{m^\beta}$ has an upper limit that can be comprised between 0 and ∞.

Let θ be this upper limit, which we will suppose finite; there will exist terms in infinite number such that

$$e^{(\theta+\varepsilon)m^\beta} > |a_m| > e^{(\theta-\varepsilon)m^\beta}.$$

Similarly, if $|a_m|$ remains smaller than 1, we could set

$$\mathbf{L}\left|\frac{1}{a_m}\right| = \frac{1}{m^\alpha},$$

$$\frac{\mathbf{LL}\left|\frac{1}{a_m}\right|}{\mathbf{L}m} = 1 - \alpha$$

will have a lower limit, for $m = \infty$, comprised between 1 and $-\infty$. Let β be this lower limit, which we suppose positive; we will have an infinite number of terms

$$e^{-m^{\beta-\varepsilon}} > |a_m| > e^{-m^{\beta+\varepsilon}},$$

and $\frac{\mathbf{L}|a_m|}{m^\beta}$ will have, for $m = \infty$, an upper limit θ that can be comprised between 0 and $-\infty$.

In these two cases, we will form $\varphi_m(t)$ for a sequence of values of m such that

$$e^{(\theta+\varepsilon)m^\beta} > |a_m| > e^{(\theta-\varepsilon)m^\beta},$$

θ being the upper limit of $\frac{L|a_m|}{m^\beta}$, which can be positive or negative. Let $A_m = e^{\theta' m^\beta}$, $\theta' < \theta$. For all terms of $\varphi_m(t)$, we have

$$\frac{1}{m} L\left|\frac{a_{m\mp\nu}}{A_m}\right| < [(\theta+\varepsilon)(1+\lambda)^\beta - \theta']m^{\beta-1},$$

and the quantity ε_m of n° 9 is of order $m^{\beta-1}$; we can, consequently, in $\varphi_m(t)$, suppose $\nu < \mathbf{H}m^{\frac{1+\beta}{2}}$. We can, moreover, remove terms such that $\mathbf{L}|a_{m\mp\nu}| - \theta' m^\beta + \frac{1}{2}Lm$ remains finite, or negative. If $\theta - \theta'$ is chosen small enough, we can thus remove all terms such that $\frac{\mathbf{L}|a_n|}{n^\beta}$ has an upper limit $\theta'' < \theta$.

Let ω be an arc variable with m, and q the number of sign changes of the real part of $a_n e^{-i\omega}$ for the terms kept in $\varphi_m(t)$. If

$$\frac{1}{m^\beta}\mathbf{L}\left|\omega_m - \omega - \frac{\pi}{2} = k\pi\right|$$

tends towards 0, as well as

$$\frac{q}{m^\beta},$$

$\frac{1}{q}\mathbf{L}\left|\frac{a_m}{A_m}\cos(\omega_m - \omega)\right|$ increases indefinitely, and the point $z = 1$ is singular.

This happens in particular if

$$\frac{1}{m^\beta}\sum_{m-\nu}^{m+\nu}|\omega_{n+1} - \omega_n|$$

tends towards 0, ω_n corresponding to the only terms kept. And as $\nu < \mathbf{H}m^{\frac{1+\beta}{2}}$, we can be sure that the point $z = 1$ is singular if, for these terms,

$$|\omega_{n'} - \omega_n| n^{\frac{1-\beta}{2}}$$

tends towards 0, a_n' and a_n being two consecutive terms.

Similarly, if

$$\frac{1}{m^\beta}\sum_{m-\nu}^{m+\nu}|\omega_{n+1} - \omega_n - \omega|$$

tends towards 0, the point $z = e^{-i\omega}$ is singular, a_n' and a_n being two consecutive terms,

$$\sum|\omega_{n+1} - \omega_n - \omega|$$

is here replaced by

$$|\omega_{n'} - \omega_n - (n-n')\omega + 2k\pi|,$$

this arc being comprised between $-\pi$ and $+\pi$.

These conditions can be fulfilled for several points and, in certain cases, for all the circle of convergence. m being always such that $\frac{\mathbf{L}|a_m|}{m^\beta}$ tends towards θ, let us suppose removed terms such that $\frac{\mathbf{L}|a_n|}{n^\beta}$ has an upper limit $\theta' < \theta$ and those such that

$$\frac{m}{\nu^2}\left[\mathbf{L}|a| - \theta' m^\beta + \mathbf{L}\frac{m}{\nu}\right]$$

remains finite, which allows us to suppose $\nu < \mathbf{H}m^{\frac{1+\beta}{2}}$. Let us divide the terms that remain in $\varphi_m(t)$ into three groups as in n° 10. We will be sure that all points of the circle of convergence are singular if $\frac{q'}{m^\beta}$ remains smaller than a quantity smaller than $\theta - \theta'$, $\frac{h}{m^\beta}$ tending towards 0, for an unlimited sequence of values of m considered. This happens, for example, if we can choose $\theta - \theta'$ small enough so that $\frac{q'}{m^\beta}$ tends towards 0.

Suppose, for example, that after having removed terms such that $\frac{\mathbf{L}|a_n|}{n^\beta}$ has an upper limit smaller than θ, we separate the others into two groups, the first of which will be of the form $\Sigma a_\nu z^{c_\nu}$,

$$\frac{c_{\nu+1} - c_\nu}{\mathbf{L}c_\nu}$$

increasing indefinitely. We will conclude that all points of the circle of convergence are singular if, for a sequence of values of m such that $\frac{\mathbf{L}|a_m|}{m^\beta}$ tends towards θ, there remain between $a_{m \mp \mathbf{H}m^{\frac{1+\beta}{2}}}$ only q' terms of the second group, $\frac{q'}{m^\beta}$ tending towards 0. θ can here represent the upper limit of $\frac{\mathbf{L}|a_n|}{n^\beta}$ for a partial sequence of values of n comprised between $m - \lambda m$ and $m + \lambda m$.

Consider, as an application, the series

$$\sum z^n e^{\omega_n i} e^{n^\beta \cos^{2n}(\pi n^\gamma)},$$

where $0 < \beta < 1$, γ being a number comprised between $\frac{1}{2}$ and 1, which we will suppose commensurable to simplify the reasoning. If n^γ is an integer,

$$\mathbf{L}|a_n| = n^\beta.$$

If we form, as in n° 10, terms such that

$$\mathbf{N}^{\frac{1}{\gamma}} + \frac{1}{4} < n < (\mathbf{N}+1)^{\frac{1}{\gamma}} - \frac{1}{4},$$

we deduce

$$\cos^{2n}(\pi n^\gamma) < (1 - \frac{1}{4}\gamma'^2 n^{2(\gamma-1)})^{2n} < e^{-\gamma'^2 n^{2\gamma-1}},$$

which tends towards 0, and for these terms $\frac{\mathbf{L}|a_n|}{n^\beta}$ has upper limit 0, for $n = \infty$; the other terms are such that

$$\mathbf{N}^{\frac{1}{\gamma}} - \frac{1}{2} < n < \mathbf{N}^{\frac{1}{\gamma}} + \frac{1}{2};$$

if $n = c_\mathbf{N}$, $\frac{c_{\mathbf{N}+1} - c_\mathbf{N}}{\mathbf{L}c_\mathbf{N}}$ increases indefinitely, and $\frac{\mathbf{L}|a_n|}{n^\beta}$ has upper limit 1. All points of the circle of convergence are therefore singular.

If $\frac{\mathbf{LL}|a_m|}{Lm}$ has upper limit 1, $\theta = 0$. We can then remove terms such that $\frac{\mathbf{LL}|a_m|}{Lm}$ has an upper limit smaller than 1, and if between a_m and a_{m+2m} there remain q terms, $\frac{q}{m}$ tending towards 0, all points of the circle of convergence are singular.

12. If $\frac{\mathbf{LL}|a_n|}{\mathbf{L}n}$ has upper limit 0, $\frac{L|a_n|}{Ln}$ will have an upper limit that can be comprised between 0 and $+\infty$. Similarly, if $|a_n|$ remains smaller than 1, $\frac{\mathbf{LL}\left|\frac{1}{a_n}\right|}{\mathbf{L}n}$ has lower limit 0, $\frac{L|a_n|}{Ln}$ could have an upper limit comprised between 0 and $-\infty$.

In general, suppose that $\frac{\mathbf{L}|a_n|}{\mathbf{L}n}$ has an upper limit β comprised between $-\infty$ and $+\infty$; so that, for a sequence of values of m, we have

$$m^{\beta+\varepsilon} > |a_m| > m^{\beta-\varepsilon}.$$

Let $\mathbf{A}_m = m^{\beta'}$, $\beta' < \beta$. By choosing $\beta - \beta'$ small enough, we could remove from $\varphi_m(t)$ all terms such that $\frac{\mathbf{L}|a_n|}{\mathbf{L}n}$ has an upper limit $\beta'' < \beta - \frac{1}{2}$. And $\frac{m}{\nu}\mathbf{L}\left|\frac{a}{\mathbf{A}_m}\cdot\frac{m}{\nu}\right|$ will remain finite for $\nu < \sqrt{\mathbf{H}m\mathbf{L}m}$, so that, in $\varphi_m(t)$, we can suppose $\nu < \sqrt{\mathbf{H}m\mathbf{L}m}$. If the real parts of $a_n e^{-i\omega t}$, for the terms that remain, have q sign changes, and if $\frac{\mathbf{L}\left|\omega_m-\omega-\frac{\pi}{2}\pm k\pi\right|}{\mathbf{L}m}$ and $\frac{q}{\mathbf{L}m}$ tend towards 0, the point $z = 1$ is singular. This happens, in particular, if $\frac{1}{\mathbf{L}m}\sum|\omega_{n+1}-\omega_n|$ tends towards 0, and, consequently, if $\frac{|\omega_{n'}-\omega_n|}{n'-n}\sqrt{\frac{n}{\mathbf{L}n}}$ tends towards 0, $a_{n'}$ and a_n being two consecutive terms kept. If $\frac{1}{\mathbf{L}m}\sum|\omega_{n+1}-\omega_n-\omega|$ tends towards 0, the point $z = e^{-i\omega}$ is singular.

Let a sequence of values of m such that $\frac{\mathbf{L}|a_m|}{Lm}$ tends towards β, β being the upper limit, for $n=\infty$, of $\frac{\mathbf{L}|a_n|}{\mathbf{L}n}$, at least when n remains comprised between $m-\lambda m$ and $m+\lambda m$. Let us suppose removed terms such that $\frac{\mathbf{L}|a_n|}{Ln}$ has an upper limit $\beta' < \beta - \frac{1}{2}$ and those such that $\frac{m}{\nu^2}\left(\mathbf{L}|a| - \beta'\mathbf{L}m + \mathbf{L}\frac{m}{\nu}\right)$ remains finite, which allows us to suppose $\nu < \sqrt{\mathbf{H}m\mathbf{L}m}$; then let us divide the terms that remain into three groups, as in n° 10. We will be sure that all points of the circle of convergence are singular if $\frac{q'}{\mathbf{L}m}$ remains smaller than a quantity smaller than $\beta-\beta'-\frac{1}{2}$, $\frac{h}{\mathbf{L}m}$ tending towards 0. This happens, in particular, if $\beta-\beta'-\frac{1}{2}$ can be chosen small enough so that $\frac{q'}{Lm}$ tends towards 0.

This is the case, for example, of the series

$$\sum z^n e^{\omega_n i} \cdot n^\beta \cos^{2n}(\pi n^\gamma),$$

where

$$\beta > \frac{1}{2}, \quad 1 > \gamma > \frac{1}{2}.$$

We can note that the number $\beta + 1$ is that which, according to M. Hadamard, gives the order of the function on the circle of convergence. And the application of the preceding method comes down to searching for singular points of maximum order, and the cases where all points of the circle of convergence are of order $\beta + 1$. We also see that this order is only finite in the case where $\frac{\mathbf{LL}|a_m|}{\mathbf{L}m}$ has upper limit 0; or if $|a_m|$ remains smaller than 1, when $\frac{\mathbf{LL}\left|\frac{1}{a_m}\right|}{\mathbf{L}m}$ has lower limit 0. In other cases, which seem to have to be considered as more general, the order on the circle of convergence is equal to $\pm\infty$.

13. A function being given, if we develop it in series following the powers of $z - a$, a being arbitrary, there will generally be only one singular point on the circle of convergence, and the analytical prolongation is possible, except in extremely particular cases; but if we suppose the coefficients a of the series $\sum a_n z^n$ given arbitrarily, with the sole condition that the radius of convergence is neither null nor infinite, the cases where analytical prolongation is impossible are much more general, and we can even wonder if the cases where the function can extend beyond the circle of convergence should not be considered as an exception. To properly specify the question, it would be necessary to make determined conventions on the way to appreciate the order of generality of the series and the coefficients a_n when n becomes very large. The results that I have obtained do not seem to me, moreover, of a nature to resolve this question definitively; but I believe I should nevertheless point them out now by showing to what extent the result seems probable.

If the radius of convergence is 1, it seems natural to consider as the most general the series in which $|a_n|$ can have, when n is very large, the largest possible variations. We will then have series for which $\frac{\mathbf{LL}|a_n|}{\mathbf{L}n}$ will have an upper limit β comprised between 0 and 1, $\frac{\mathbf{L}|a_n|}{n^\beta}$ having upper limit θ for limit.

Suppose separated the terms such that $\frac{\mathbf{L}|a_n|}{n^\beta}$ remains greater than $\theta' < \theta$; it seems natural to consider as the most general the case where the number of terms of this group, comprised between a_n and $a_{n'}$, is infinitely small compared to $n' - n$. We have seen that all points of the circle of convergence are singular in the case where, between $a_{n-\mathbf{H}n^{\frac{1+\beta}{2}}}$ and $a_{n+\mathbf{H}n^{\frac{1+\beta}{2}}}$, the terms of this group can be divided into two, some of the form $\Sigma a_\nu z^{c_\nu}$, $\frac{c_{\nu+1}-c_\nu}{\mathbf{L}c_\nu}$ increasing indefinitely, whose number can be, in this interval, $\frac{\varepsilon}{\mathbf{L}n} \cdot n^{\frac{1+\beta}{2}}$, the others in number εn^β.

The series thus obtained are already very general, without being able to consider them as the most general. But they have been formed by comparing the value of $\varphi_m(t)$ at a point anywhere on the circle of convergence to $e^{\theta m^\beta}$. We could form series such that $\varphi_m(t)$ has at each point of this circle a value of a variable order of magnitude, which would be even more general.

In summary, we come to form much more general series than those already known, which cannot be prolonged analytically; and there is reason to think that we can form even more general ones. Consequently, without being able to affirm definitively, it seems probable that the most general series are those whose analytical prolongation is impossible.